

# A Bethe lattice representation for sandpiles

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Avalanches in sandpiles are represented throughout a process of percolation in a Bethe lattice with a feedback mechanism. The results indicate that the frequency spectrum and probability distribution of avalanches resemble more to experimental results than other models using cellular automata simulations. Apparent discrepancies between experiments are reconciled. Critical behavior is here expressed throughout the critical properties of percolation phenomena.

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## INTRODUCTION

The idea of self-organized criticality (SOC) proposed by Bak, Tang and Wiesenfeld [1,2] triggered a lot of experimental as well as theoretical work on relaxation processes in granular materials. Sandpiles seem to be the simplest systems to test self-organized behavior. The importance of its study comes from the fact that SOC has been suggested as a possible explanation for the power law behavior seen in many systems: earthquakes, [3] mass distribution in the Universe, star flickers, etc. [4]

Experiments on sandpiles were designed and performed in [5,6]. In [5], avalanche sizes were recorded in rotating drum experiments, finding that avalanches, instead of being distributed over all sizes obeying a power law distribution as predicted in [1,2], occurred quite regularly in size and time, in an almost periodic pattern (See also [7,8]).

In [6], mass fluctuations in an evolving sandpile were studied, showing that for small enough sandpiles, the observed mass fluctuations are scale invariant, and probability distribution of avalanches shows finite size scaling whereas large sandpiles do not. In this experiment, small sandpiles seemed to exhibit SOC. Besides, an apparent disagreement has emerged between the results reported in [5] and [6] but, as we will show in this paper, these results are essentially the same.

Though many other theoretical and experimental works were performed [9–28], some of the later proposed models were devoted to the problem of SOC in a more general fashion than the sole application to sandpiles (e.g. [9,10,12,16]). Some others fix their attention in models for which particular mechanisms of interaction seem to be relevant [13–15,18,21,28].

In the present work we propose a representation of the avalanche process in sandpiles as a percolation in a Bethe lattice, capturing the essential features of the avalanche phenomenon and simultaneously taking into account the nature of the sandpile, in order to reproduce the experimental results.

The image of an avalanche as an initial object that consecutively drags another resembles a branching process for which the Bethe lattice representation seems to be natural. This branching process was proposed in [22], showing good possibilities to describe the change of behavior of fragment size distribution in fragmentation phenomena. Another branching process representation was proposed in [12,26] in an attempt to obtain analytical solutions for avalanche processes, and in [23,24], a self-organizing branching process was proposed with a closer relation with properties of physical systems. There, a feedback mechanism is introduced, but the branchig structure is not related to the physical nature of the system.

The introduction of a Bethe lattice representation, as will be seen, has the advantage of its generality because the process of dragging is characterized by a *drag probability*  $p$  for each one of the particles forming the nodes of the lattice. The nature of the system is taken into account through the relation of  $p$  with the parameters characterizing the self-organizing characteristics of the system, i.e. the slope angle  $\theta$  and the size (number of grains) of the pile,  $N$ . This viewpoint is similar to that of [23,24] but with a much closer relation with the experiments and the physical nature of the studied systems, i.e., sandpiles. The main features obtained there can be reproduced with this representation, but we will focus on the polemics related with experimental data [7].

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In section 2 we describe the representation of the avalanches and expose the relation between the drag probability and sandpile characteristics.

In section 3, the results of simulations with this representation are exposed and compared with experimental results. Section 4 is devoted to the conclusions.

## REPRESENTING THE SANDPILE AND THE AVALANCHES

The phenomena for which sandpiles seem to be the simplest paradigm should manifest themselves in a representation independent of the avalanche mechanisms. A representation through a Bethe lattice resembles well the process, and to reveal the behavior of the phenomenon, only the essential characteristics should be involved.

Let us represent the avalanche as a cascade process in the Bethe lattice as follows: Firstly, we start with a single node, which could represent in this case a grain. With probability  $p$ , from this node will emerge  $F$  new nodes, representing that the initial particle has generated  $F - 1$  new indistinguishable particles, which in principle will continue the avalanche. This operation (generation of new identical particles with probability  $p$ ) is applied to each node of this new group. In this case some of them will continue the generation, some will not, and so on. The process is represented in fig.1 for  $F = 3$ . Empty nodes represent those in which the process of percolation in this lattice will not progress, mimicking those particles that do not follow the falling process in the cascade. This representation seems to be natural for the process and does not appeal to the nature of the forces between the grains in the sandpile, the nature of the grains or even the nature of the avalanche.

Once the percolation process overcomes a given length (that of the border of the sandpile), those nodes beyond the limit constitute the avalanche. Counting the number of grains that just surpass the length of the pile is equivalent to measuring the size of the avalanche. Knowing the number of grains in the pile, it is possible to register the mass variations in the pile with time,  $M(t)$ . The percolation process stops once the limit is surpassed; the pile *reorganizes itself* with the new number of grains, (i.e. a new slope is calculated with the remaining grains) and again starts growing until a new avalanche develops. This establishes a feedback mechanism in this model, linking it with the experimental conditions.

We now relate the percolation probability with the parameters characterizing the pile, namely, its slope and size, using the simplest representation. To do this let us represent a conic sandpile of height  $h$ , radius  $R$  and base slope  $\theta$ . A small sphere of "effective radius"  $r_0$  can characterize the size of the grain of sand. The ratio of the volume of the pile to that of the grain (including porosity effects in the value of  $r_0$ ) gives the number of grains in the pile:

$$N = \frac{1}{4}x^3 \tan \theta$$

where  $x = R/r_0$  is the ratio of the radius of the pile and the effective radius of the grain.

The percolation probability is in this case translated to a *drag probability*  $p(\theta)$  of one grain to the next  $F - 1$  situated down the slope. In this way, the interpretation of fig.1 is straightforward.

The dependence of  $p$  with  $\theta$  can be formulated *a priori* taking into account that the drag probability should be an increasing function of the slope. A good variable to describe this slope seems to be  $\tan \theta$ . On the other hand, the drag probability should be small for angles less than the critical, and, once surpassed that angle, the probability for an avalanche to take place must increase sharply. Let us propose the following dependence:

$$p(\theta, T) = \frac{\exp(\frac{\tan \theta - \tan \theta_c}{T})}{\exp(\frac{\tan \theta - \tan \theta_c}{T}) + 1} \quad (1)$$

where  $\theta_c$  is the critical angle, related with the Coulomb's law  $\mu = \tan \theta_c$  ( $\mu$  is the friction coefficient and here, for simplicity, will be taken as unity), and  $T$  is a parameter through which we may control the sharpness of the variation just at  $\theta = \theta_c$  and can be used to include factors like granularity and vibrations [27]. Using the relation between the slope angle and the number of grains in the pile, in this case (1) can be expressed as:

$$p(y, T) = \frac{1}{1 + \exp(\frac{1-y}{T})} \quad (2)$$

where  $y = N/N_c$ .  $N_c$  is the number of grains corresponding to  $\theta_c$ . This is valid for small  $T$  so that the variation of  $p(y, T)$  is sharp near  $y = 1$ , i.e.  $T \ll 1/4$ .

Once the pile arrives to a size around  $N_c$  the avalanches will be noticeable and the slope will be readjusted after each avalanche. The process of adding grains will again vary the value of  $p$  up to values near  $p_c$ , -the value of the

probability corresponding to  $\theta_c$ -, to produce another avalanche, and so on. This constitutes a mechanism of feedback, since the flux of sand tends to be kept constant because of the concurrence of sand supply and avalanches. This has been recognized as necessary [25] to deal with critical behavior. Because of size effects, avalanches will be registered for  $p$  slightly less than  $p_c$ .

If, for a given value of  $p$ , an avalanche develops, it will be counted if the number of steps in the Bethe lattice surpasses the length of the slope of the pile. The length of the slope,  $L$ , measured in units of the grain diameter, is  $g = x/2\sqrt{1-p}$ , and this is the threshold for an avalanche to be registered. When the value  $g$  is surpassed, those grains (nodes in the Bethe lattice) belonging to that generation, are counted as the size of the avalanche. This will permit to relate the results of the simulations with the measured magnitudes, namely, the size of the avalanches,  $S(t)$  [5] or the mass of the piles,  $M(t)$  [6], as a function of time. In this last case, the mass of the sandpile is represented as the number of grains  $N(t)$ .

## RESULTS OF THE SIMULATIONS

Simulations were performed for a wide range of values of  $x$ , ranging from 10 to 500, using (2) for  $T = 0.1$  and a Bethe lattice with two branches. For each value of  $x$ , more than  $2^{16} * 100$  realizations were performed. Collected data were  $N(t)$  and  $S(t)$ .

The temporal fluctuations of the mass  $N(t)$ , measured as the number of grains in the pile, in units of  $N_c$ , are plotted in figure 2 (a), (b) and (c) for  $x = 50, 100$  and  $500$ . The unit of time used was that between two consecutive events of adding grains. Avalanches are considered as instantaneous. This behavior resembles that reported in [6]. Note that different time scales were used in fig. 2 (a), (b) and (c) for a better illustration of the time variation for different sizes.

Figure 3 shows the probability of avalanche sizes  $P(s)$  scaled as  $x^{1.9}$ , as in [6], vs  $s$  normalized to  $x^{0.95}$  for  $x = 50$  and  $500$ , showing a good finite size scaling  $P(s)x^\beta$  vs  $s/x^\nu$  with  $\beta = 2\nu$  as in [6]. The exponent  $\nu = 0.95$  was chosen as the best fitting for all data with  $s/x^\nu < 2$ . This representation reproduces the finite size scaling for all sizes of sandpiles. The exponential falloff can be verified. Larger piles show more dispersion for large avalanche sizes. This can be explained noting that in the percolation process, large percolation lengths correspond to probabilities near the critical one, where critical behavior dominates and fluctuations are stronger.

Figure 4 (a) and (b) shows the power spectrum of  $N(t)$  and  $s(t)$  respectively for  $x = 50, 100$  and  $500$  exhibiting in (a) a clear  $1/f^2$  dependence, which also coincides with the results of [6]. Power spectrum in (a) reveals the same characteristics for all sandpiles, i.e. dependence  $1/f^\beta$  with  $\beta \simeq 2$ , whereas in (b), the power spectrum is clearly flat as in [5].

Concerning the apparent disagreement between the experimental results in [5] and [6], it must be said that the process of measure in both experiments is essentially different, since in [6] the mass of the pile is recorded as a function of time, whereas in [5] the experiments record the variation of avalanche size as a function of time, i.e., the magnitude that would correspond to the temporal variation of the number of grains in the rotating cylinder, that means its time derivative. This leads to a different power spectrum, so that if the power spectrum obtained in [6] is  $1/f^2$ , in this case its derivative should have a "flat" spectrum, in correspondence with the results of [5], and there is no disagreement. In our case the spectrum is flat for high frequencies because we are considering the avalanches as instantaneous, but this is not an essential point. The simulations could be improved introducing a finite time for avalanches. Though both teams have argued about the differences of their experimental setups, we think that our argument shows a very important difference.

The sizes of the avalanches  $S(t)$  represent the value of the variation of the function  $N(t)$  in each jump, i.e., the derivative of that function, so that the spectra in [5] and [6], though could be interpreted as expressions of different behavior, are really intimately related.

The particular dependence of the drag probability with the slope is not of great importance for the main results of this work. Dependences as  $\sin \theta$ ,  $\sin^2 \theta$  and others can be used in the simulations without significant changes in the results concerning probability distribution of avalanches, power spectrum, etc. The main property required is the increase of the drag probability with the slope. Also, the number of terminals in the Bethe lattice is not important for the main conclusions. This reveals the robustness of this phenomenon.

## CONCLUSIONS

A Bethe lattice representation linked with characteristics of sandpiles, including a feedback mechanism, has been presented in the same direction outlined in [23] with a new viewpoint, closely related with the physical nature of the sandpile, which leads us to a closer link with the experiments. The reproducibility of the experimental results lies on

the fact that the Bethe lattice representation unravels the physical nature of the avalanche process in sandpiles and is able to be linked with the geometrical properties of the system.

This representation keeps the same nature for all avalanches, irrespective of pile size. In the avalanche process it seems to exist some kind of transition, manifested in the change of behavior of the size distribution of avalanches when the piles are large, reflected in the increase of fluctuations in the region of large avalanches. This reflects that the process of percolation in the Bethe lattice approaches the critical point so that the second order nature of the phenomenon reveals itself in the language of percolation phenomena. In this way, the description of avalanches has been translated to the problem of percolation in a Bethe lattice, and in this sense the phenomenon is critical. Thus, SOC, examined with this viewpoint, is present in the organization of avalanches.

The proposed representation seems to reproduce the behavior of the sandpiles, is very simple to instrument in a computer, and reveals an essentially unique behavior in small and large sandpiles. Besides, it contains the main characteristics of sandpiles in the sense of the increase of avalanche probability by adding sand grains and a readjustment of the slope after each avalanche. Oscillations of  $p$  and  $\theta$  near a critical value are properties of this model as they are also in the branching process model proposed in [23,24]. Finite size scaling for different sizes was obtained for the distribution of avalanches with good reproduction of the experimental behavior.

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FIG. 1. Bethe lattice representation of the sandpile behavior.

FIG. 2. Fluctuations of the mass of the sandpile  $N(t)/N_c$  for (a)  $x = 50$ , (b)  $x = 100$  and (c)  $x = 500$  obtained from simulations in a Bethe lattice as described in the text. The behavior for other number of terminals in the lattice, or another plausible dependence of  $p(\theta)$  is essentially the same.

FIG. 3. Avalanche probability scaled as  $x^{1.9}$  vs. avalanche size  $s$  normalized to  $x^{.95}$  for  $x = 50$  (circles) and 500 (squares). The result for this theoretical sandpile resemble those of [6] with exponential falloff. Large piles show larger dispersion for large avalanches.

FIG. 4. (a).- Power spectrum of mass fluctuations for different sizes of the sandpiles  $x = 50, 100$  and 500 as indicated in the figure. The spectrum is  $1/f^2$  in agreement with [6]. then the corresponding spectrum for the avalanches is flat as in [5]. (b).- Power spectrum of the avalanche sizes for the same set of values as in (a).











